LECTURE 3a

Design Problem Manipulations and Solution Strategies I

The mappings used in the real design problems are almost always such that inverse functions r_i^{-1} for the inverse mapping $\mathbf{x}^A = r_i^{-1}(\mathbf{l}^A)$ do not exist and that only so called inverse image can be constructed. Basic problems are :

- x_i is often discrete or integer,
- subjective value function may not be consistent or even exist,
- criteria sets contain non-linear functions or even procedures,
- factors w have multiple values since they are subjective to all involved in design process,
- feasible domain $\mathbf{X}^{\geq}$ is non-convex and often multiply connected (vibration problems),
- no insight in design space is provided.

Classifications following Rao are given in the sequel.

METHODS OF OPERATIONAL RESEARCH

Mathematical programming techniques

- Calculus methods
- Calculus of variations
- Nonlinear programming
- Geometric programming
- Quadratic programming
- Linear programming
- Dynamic programming
- Integer programming
- Stochastic programming
- Separable programming
- Multiobjective programming
- Network methods: CPM and PERT
- Game theory

Stochastic process techniques

- Statistical decision theory
- Markov processes
- Queuing theory
- Renewal theory
- Simulation methods
- Reliability theory

Statistical methods

- Regression analysis
- Cluster analysis, Pattern recognition
- Design of experiments
- Discrimainate analysis (Factor analysis)

Application of scientific techniques to DM problems to establish optimal solutions (minimum; prob. distribution; model represent.)

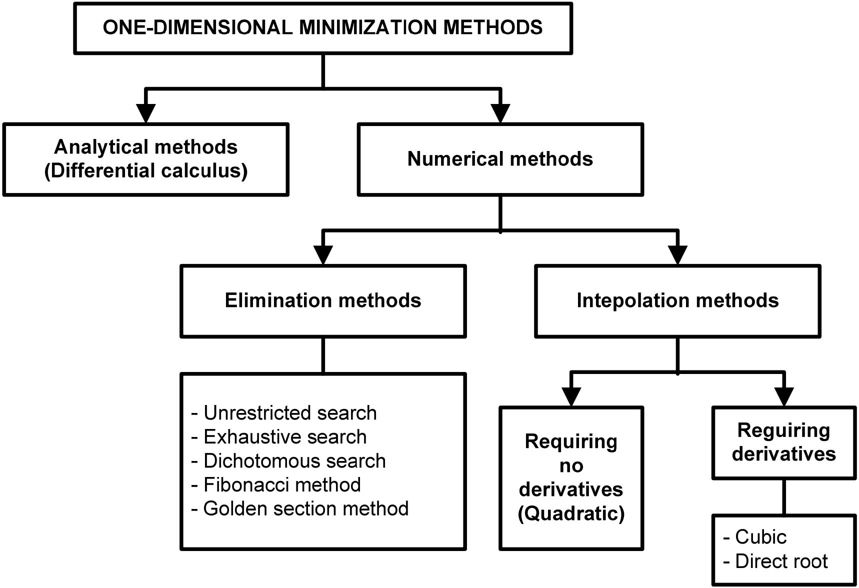
UNCONSTRAINED MINIMIZATION METHODS

Direct search methods
(Do not require the derivatives of the function)

- Random search method
- Univariate method
- Pattern search method
  - (a) Powell's method
  - (b) Hooke & Jeeves method
- Rosenbrock's method of rotating coordinates
- Simplex method

Descent methods
(Require the derivatives of the function)

- Steepest descent method
- Conjugate gradient method (Fletcher-Reeves)
- Newton's method
- Variable metric method (Davidon-Fletcher-Powell)


ONE-DIMENSIONAL MINIMIZATION METHODS
Analytical methods
(Differential calculus)
Numerical methods
Elimination methods
Intepolation methods
- Unrestricted search
- Exhaustive search
- Dichotomous search
- Fibonacci method
- Golden section method
Requiring no derivatives (Quadratic)
Reguiring derivatives
- Cubic
- Direct root

CONSTRAINED OPTIMIZATION TECHNIQUES

Direct methods

- Heuristic search methods - the complex method
- Constraint approximation methods – the cutting plane method
- Methods of feasible directions
  (a) Zoutendijk's method
  (b) Gradient projection method

Indirect methods

- By the transformation of variables
- Penalty function methods
  (a) Interior penalty function method
  (b) Exterior penalty function method

Basic phases in problem definition process

1. **Problem identification (x, p, Y, G):**

   (a) **Independance** of parameters;

   (b) **Diagonalization** of the design matrix **DR**= **DM * DP**;

   (c) **Minimization of informaton content**.

Basic phases in problem definition process

2. **Problem reduction**:
   (a) **Elimination of constant influences** in f(x),
   (b) **Regionalization** (Problem separability);
   (c) **Decomposition**:
   - Model coordination;
   - Goal coordinaton.

3. **Manipulation** into mathematical model that is easier to solve.

4. Choice of **strategy for problem solution** (Including iterations).

Problem classification

1. Classification according to constraints:

   (a) **Constrained** problem; (b) **Unconstrained** problem.

2. Classification according to the nature of the design variables:

   (a) Static (parametric) problem - determine **X**;

   (b) Dynamic (trajectory) problem - determine function X(t).

3. Classification according to the phisical structure of the problem:

   (a) Problems with optimal control ;
   (b) Problems with non-optimal control.

   Controls: they govern the development of the system;
   State variables: they are describing the state of the system;

Problem classification

4. Classification according to the type of equations:

   (a) Linear problems; f( ), g( ), h( ) linear

   (b) Non-linear p.; f( ) or g( ) or h( ) non-linear

   (c) Geometrical p.: min $f(x) = \Sigma\, c_i \prod x_j^{pij}$; $\mathbf{c} > 0$; $\mathbf{x} > 0$;

   (d) Quadratic p.: $f(\mathbf{x}) = c + \mathbf{q}^T\mathbf{x} + \mathbf{x}^T\mathbf{Q}\mathbf{x}$**;** $g(\mathbf{x})$, $h(\mathbf{x})$ linear

5. Classification according to the allowable values of d. variables:

   (a) Integer programming discrete variables / binary variables (0,1);

   (b) Real values (continuous variables).

Problem classification

6. Classification according to character of the variables:

   (a) Deterministic programming;

   (b) Stochastic programming.

7. Classification according to the separability of the functions:

   (a) Separable problem $f(x) = \Sigma f_i(x_i)$; $\Sigma g_{ji}(x_i) < b_j$;
   (b) Non-separable problem

8. Classification according to the number of objective functions:
   (a) Unicriterial;
   (b) Multicriterial: - MADM;
   - MODM.

M C D M
Multiple Criteria Decision Making

- **MADM**

  Multiple Attribute Decision Making

  selection among discrete no. of designs

- **MODM**

  Multiple Objective Decision Making

  optimisation over infinite no.of designs

  nonlinear problem over nonconvex, multiply connected domain ?!

Design generation and evaluation in affine space

Design selection in metric space

MODM MANIPULATIONS

Extreme of A Function With A Subsidiary Condition (Washizu)

PROBLEM: Find the extreme of the function:

$$z = f(x,y) = x^2 + y^2 -2x -4y +6 \qquad (1)$$

under the subsidiary condition:

$$g(x,y) = 2x + y -1 = 0 \qquad (2)$$

Geometrically speaking, the problem is one of finding the extremal value of z on the curve of intersection of $z = f(x, y)$ and $g(x, y) = 0$.

Extreme of A Function With A Subsidiary Condition

(A) One of the ways of solving the problem is to eliminate one of the variables, say *y*, from Eq. (1) by the use of Eq. (2), thus obtaining

$$z = f(x,y(x)) = f^*(x) = 5x^2 + 2x + 3 \qquad (3)$$

and then finding the extreme of z by the condition

$$df^*/dx = 10x + 2 = 0 \qquad (4)$$

By solving Eq. (4), we obtain

$$x = -1/5 \qquad (5)$$

and find that

$$z_{ext} = 14/5, \quad y = 7/5 \qquad (6)$$

It is observed that this external value proves to be the absolute **minimum** of $f^*(x)$.

Extreme of A Function With A Subsidiary Condition

(B) The method of ***Lagrange multipliers*** *asserts that the above problem is equivalent to finding the stationary value of the* ***Lagrange function L ≡ z****, defined by:*

$$z_I = x^2 + y^2 - 2x - 4y + 6 + \lambda(2x + y - 1) \qquad (7)$$

where the independent variables are x, y and λ and the stationary conditions are given by

$$\partial z_I/\partial x = 2x - 2 + 2\lambda = 0 \qquad (8)$$

$$\partial z_I/\partial y = 2y - 4 + \lambda = 0 \qquad (9)$$

$$\partial z_I/\partial \lambda = 2x + y - 1 = 0 \qquad (10)$$

By solving these equations, we obtain

$$x = -1/5, \quad y = 7/5, \quad \lambda = 6/5 \qquad (11)$$

and find the stationary value of z_I as follows:

$$z_{Ist} = 14/5 = z_{ext} \qquad (12)$$

Extreme of A Function With A Subsidiary Condition

(C) If one of the independent variables, say x, is eliminated from z_I through the use of Eq. (8), the function is transformed into

$$z_{II} = y2 + \lambda y - \lambda^2 -4y +5 + \lambda \quad (13)$$

where there remain only two independent variables, namely λ and y. The stationary conditions are then given by:

$$\partial z_{II}/\partial y = 2y - 4 + \lambda = 0 \quad (14)$$

$$\partial z_{II}/\partial \lambda = y + 1 - 2\lambda = 0 \quad (15)$$

yielding immediately

$$y = 7/5, \quad \lambda = 6/5 \quad (16)$$

and consequently

$$z_{IIst} = 14/5 = z_{ext}, \quad x = -1/5 \quad (17)$$

Extreme of A Function With A Subsidiary Condition

(D) Going further, we shall eliminate x and y from z_I through the use of Eqs. (8) and (9). Then, the function is transformed into

$$z_{III} = -5/4\lambda^2 + 3\lambda + 1 \quad (18)$$

where λ is the only remaining independent variable. The stationary condition is then given by

$$d\, z_{III}/d\lambda = -5/2\lambda + 3 = 0 \qquad (19)$$

which gives

$$\lambda = 6/5 \qquad (20)$$

and consequently

$$z_{IIIst} = 14/5 = z_{ext}, \quad x = -1/5, \quad y = 7/5 \qquad (21)$$

It is observed that z_{IIIst} is the absolute **maximum** of the function z_{III} with respect to the single variable λ.

Extreme of A Function With A Subsidiary Condition

Thus, it has been shown that the stationary values are the same for all the transformed formulations.

The extreme value obtained as the minimum in the function (3) is given as the maximum in the function (18).

However, z_{Ist} and z_{IIst} are no longer either maximum or minimum of the functions z_I, and z_{II}, respectively.

Extreme of A Function With A Subsidiary Condition (Linear problem)

$\mathbf{c}^{\mathrm{T}}$ ... coefficients' vector of the objective function $f(\mathbf{x}) = \mathbf{c}^{\mathrm{T}}\mathbf{x}$

$\mathbf{A}$ ... coefficients' matrix of the constraints $g_i(\mathbf{x}) = \mathbf{a}_{row\ i}\ \mathbf{x} - \mathrm{b}_i \le 0$

Matrix dimension: number of the constraints * number of variables.

$\mathbf{b}$ ... vector of the RHS of constraints (j = 1,…ng)

$\mathbf{x}$ ... vector of the primal variables (i = 1,…nv)

$\boldsymbol{\mu}$ ... vector of the multipliers -dual variables (j = 1,…ng)

Primal problem reads:

$$\min_{\mathbf{x}} f(\mathbf{x}) = \mathbf{c}^{\mathrm{T}}\mathbf{x}$$

Such that: $\mathbf{A}\ \mathbf{x} \le \mathbf{b}, \quad \mathbf{x} \ge 0$

Lagrange function:

$$L(\mathbf{x}, \boldsymbol{\mu}) = \mathbf{c}^{\mathrm{T}}\mathbf{x} + \boldsymbol{\mu}^{\mathrm{T}} \cdot (\mathbf{A}\ \mathbf{x} - \mathbf{b}) = (\mathbf{c}^{\mathrm{T}} + \boldsymbol{\mu}^{\mathrm{T}}\mathbf{A})\mathbf{x} - \boldsymbol{\mu}^{\mathrm{T}}\mathbf{b}\ ;\ \boldsymbol{\mu} \ge 0$$

Extreme of A Function With A Subsidiary Condition (Linear problem)

The finite minimum exists only for satisfied the optimality conditions:

$$\left(\mathbf{c}^{\mathrm{T}} + \boldsymbol{\mu}^{\mathrm{T}}\mathbf{A}\right) \geq 0,$$

because, if $\left(\mathbf{c}^{\mathrm{T}} + \boldsymbol{\mu}^{\mathrm{T}}\mathbf{A}\right) < 0$ the minimum tends to $-\infty$.

Furthermore, it follows that $x_i = 0$ if $c_i + \mu_k A_{k_i} \neq 0$ that is:

$$\left(\mathbf{c}^{\mathrm{T}} + \boldsymbol{\mu}^{\mathrm{T}}\mathbf{A}\right)\mathbf{x} = 0$$

since that term increases the value of the function $L(\mathbf{x}, \boldsymbol{\mu})$.

Extreme of A Function With A Subsidiary Condition (Linear problem)

The dual function is obtained by minimization of L over $\boldsymbol{x}$.

It reads:

$$d(\boldsymbol{\mu}) = -\mathbf{b}^{\mathrm{T}}\boldsymbol{\mu}$$

$$\boldsymbol{\mu} \in D; \quad D = \left\{ \boldsymbol{\mu} \middle| \mathbf{A}^{\mathrm{T}}\boldsymbol{\mu} + \mathbf{c} \geq 0;\ \boldsymbol{\mu} \geq 0 \right\}$$

Accordingly the **dual problem** reads:

$$\max_{\boldsymbol{\mu}} d(\boldsymbol{\mu}) = -\mathbf{b}^{\mathrm{T}}\boldsymbol{\mu}$$

Such that: $\quad \mathbf{A}^{\mathrm{T}}\boldsymbol{\mu} + \mathbf{c} \geq 0 ; \qquad \boldsymbol{\mu} \geq 0$

Dual function

$$d(\boldsymbol{\lambda},\boldsymbol{\mu}) = \min_{x \in S_1;\ \boldsymbol{\lambda},\boldsymbol{\mu} \in D} L(x,\boldsymbol{\lambda},\boldsymbol{\mu})$$

Where: $D = \left\{ \boldsymbol{\lambda},\boldsymbol{\mu} \middle| \exists\, d(\boldsymbol{\lambda},\boldsymbol{\mu}),\ \boldsymbol{\mu} \geq 0 \right\}$

Prove that: $d(\boldsymbol{\lambda},\boldsymbol{\mu}) \leq f(\mathbf{x})$ for $\forall \mathbf{x} \in S$ and $(\boldsymbol{\lambda},\boldsymbol{\mu}) \in D$

Since: $d(\boldsymbol{\lambda},\boldsymbol{\mu}) = \min_{x \in S_1 (x_i > 0)} L(\mathbf{x},\boldsymbol{\lambda},\boldsymbol{\mu}) \leq L(\mathbf{x},\boldsymbol{\lambda},\boldsymbol{\mu})$

that is for $\mathbf{x} \in S = S_1 \cap S_2 \cap S_3$:

$$d(\boldsymbol{\lambda},\boldsymbol{\mu}) \leq f(\mathbf{x}) + \underbrace{\boldsymbol{\lambda}^{\mathrm{T}} \cdot \mathbf{e}(\mathbf{x})}_{\mathbf{x} \in S_2 (e_i = 0)} + \underbrace{\boldsymbol{\mu}^{\mathrm{T}} \cdot \mathbf{g}(\mathbf{x})}_{\mathbf{x} \in S_3 (g_j \leq 0)}$$

At optimum (denoted $^{\text{o}}$)

$$d\left(\boldsymbol{\lambda}^{\text{O}},\boldsymbol{\mu}^{\text{O}}\right)=f\left(\mathbf{x}^{\text{O}}\right)+\underbrace{\boldsymbol{\lambda}^{\text{OT}}\cdot\mathbf{e}\left(\mathbf{x}^{\text{O}}\right)}_{=0}+\underbrace{\boldsymbol{\mu}^{\text{OT}}\cdot\mathbf{g}\left(\mathbf{x}^{\text{O}}\right)}_{=0}$$

$$d\left(\boldsymbol{\lambda}^{\text{O}},\boldsymbol{\mu}^{\text{O}}\right)=f\left(\mathbf{x}^{\text{O}}\right)$$

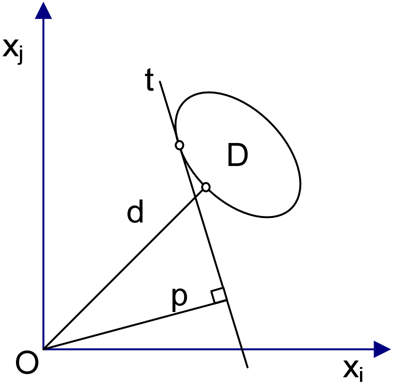
Since $d\left(\boldsymbol{\lambda},\boldsymbol{\mu}\right)<f\left(\mathbf{x}\right)$ it follows that $d\left(\boldsymbol{\lambda}^{\text{O}},\boldsymbol{\mu}^{\text{O}}\right)$ is the maximum of the function d .

The maximization of the function d over a domain is called dual problem which gives $\boldsymbol{\lambda}^{\text{O}},\boldsymbol{\mu}^{\text{O}}$:

$$\max_{\boldsymbol{\lambda},\boldsymbol{\mu}} d\left(\boldsymbol{\lambda},\boldsymbol{\mu}\right),$$

such that $\left(\boldsymbol{\lambda},\boldsymbol{\mu}\right)\in D$

Considerable advantages if ng > nv (almost always)



Primal - dual relationship:

Minimal distance d from origin O to domain D equals maximal normal distance p from origin to the tangent t separating origin O and domain D, i.e.

$$\min d = \max p$$

Khun - Tucker conditions (see Bletzinger)

Where $L(x,\lambda,\mu) = f(x) + \boldsymbol{\lambda}^{\mathrm{T}} g_j(x) + \boldsymbol{\mu}^{\mathrm{T}} e_j(x)$ is Langrangian, with λ and μ the vectors of Langrange multipliers.

Stationary conditions read:

$$L,_{x_i} = f,_{x_i} + \boldsymbol{\lambda}^{\mathrm{T}} \nabla \mathbf{g} + \boldsymbol{\mu}^{\mathrm{T}} \nabla \mathbf{e} = 0; \qquad i = 1,\dots,n$$

$$L,_{\lambda_j} = g_j(x) = 0; \qquad j = 1,\dots,p$$

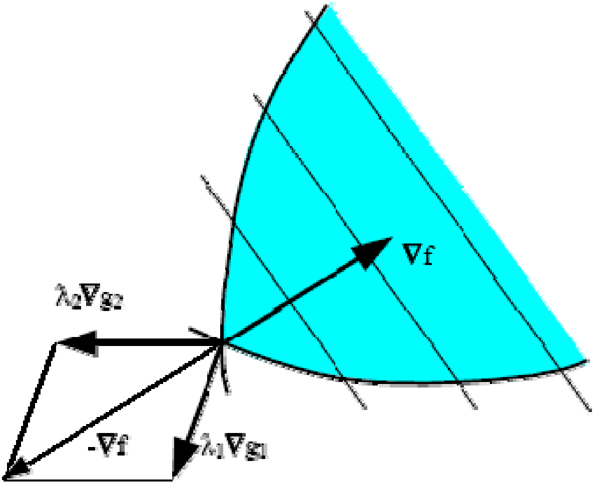
$$L,_{\mu_j} = e_j(x) = 0; \qquad j = 1,\dots,q$$

$$\lambda_j g_j = 0; \qquad j = 1,\dots,p$$

$$\lambda_j \geq 0; \qquad \lambda_j \geq 0$$

while last two conditions supply criterion to select active inequality constraints (g = 0 and λ>0 or g<0 and λ=0).

Graphical representation of the Kuhn-Tucker conditions.



Some practical methods

Optimization problems can be stated in the most general form as:

$$\begin{aligned} &\text{minimize} && f(x);\; x \in \mathbf{R}^{\mathrm{n}} \\ &\text{such that} && g_j(x) \le 0; \qquad j = 1,\dots,p \\ & && e_j(x) = 0; \qquad j = 1,\dots,q \end{aligned}$$

or

$$\text{minimize } \left\{ f(x) \middle| \mathbf{g}(x) \le 0,\ \mathbf{e}(x) = 0;\ \mathbf{g}:\mathbf{R}^{\mathrm{n}} \to \mathbf{R}^{\mathrm{p}},\ \mathbf{e}:\mathbf{R}^{\mathrm{n}} \to \mathbf{R}^{\mathrm{q}} \right\}$$

1 Quadratic programming

$$\begin{aligned} &\text{minimize} && f(x) = \frac{1}{2}\mathbf{x}^{\mathrm{T}}\mathbf{Q}\mathbf{x} + \mathbf{c}^{\mathrm{T}}\mathbf{x} \\ &\text{subject to} && g(x) = \mathbf{A}\mathbf{x} + \mathbf{b} \le 0 \\ & && e(x) = \mathbf{C}\mathbf{x} + \mathbf{d} \le 0 \end{aligned}$$

2 Linear programming

$$\begin{aligned} &\text{minimize} && f(x) = \mathbf{c}^{\mathrm{T}}\mathbf{x} \\ &\text{subject to} && g(x) = \mathbf{A}\mathbf{x} + \mathbf{b} \le 0 \\ & && e(x) = \mathbf{C}\mathbf{x} + \mathbf{d} \le 0 \end{aligned}$$

Penalty methods

$$f(\mathbf{x},\mathbf{c}) = f(\mathbf{x}) + c\sum P(g(\mathbf{x}))$$

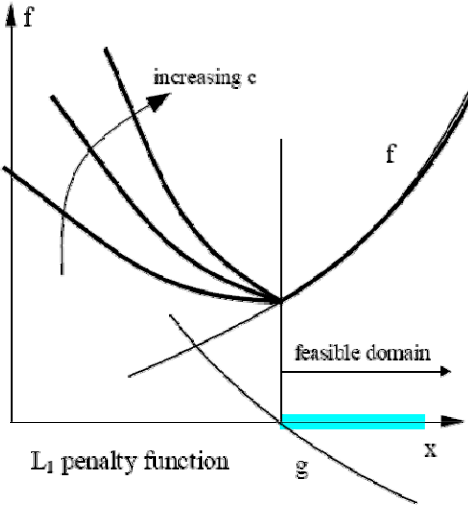
3. Exterior penalty method

$$g_j^+(\mathbf{x}) = e_j(\mathbf{x}); \qquad j = 1, \ldots, q \quad (\text{all equality constraints})$$

$$g_j^+(\mathbf{x}) = \max[0, g_j(\mathbf{x})]; \qquad j = 1, \ldots, p \quad (\text{all active inequality constraints})$$

	name	formula	comments
(a)	L_1 - penalty method also called "exact" penalty method	$P(\mathbf{x}) = \sum_j \left\|g_j^+(\mathbf{x})\right\|$	advantage: optimum can be found exactly with finite penalty factor disadvantage: discontinuous gradient at the boundary of feasible domain
(b)	Quadratic penalty function	$P(\mathbf{x}) = \sum_j \left(g_j^{\vert}(\mathbf{x})\right)^2$	advantage: continuous gradients reduces numerical problems disadvantage: optimum of modified problem always infeasible

Exterior penalty function method



L_1 penalty function

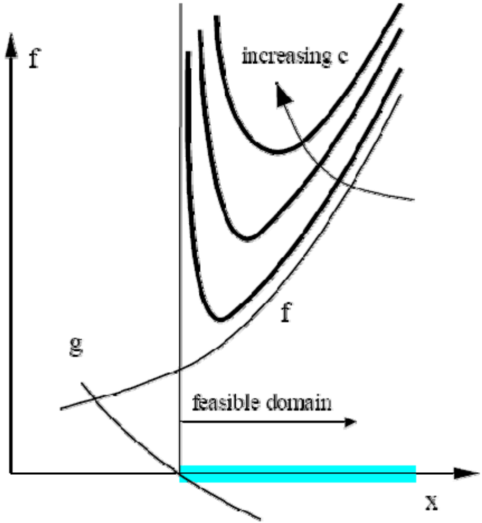
4. Interior penalty or barrier function method

$$f(\mathbf{x},\mathbf{c}) = f(\mathbf{x}) + \frac{1}{c}\sum B\left(g(\mathbf{x})\right)$$

(a) $$B(\mathbf{x}) = \sum_{j=1}^{p} \frac{1}{-g_j(\mathbf{x})}$$

(b) $$B(\mathbf{x}) = \sum_{j=1}^{p} \frac{1}{\left(-g_j(\mathbf{x})\right)^{\varepsilon}};$$ $\varepsilon > 0$, preferably $\varepsilon = 2$

(c) $$B(\mathbf{x}) = -\sum_{j=1}^{p} \ln\left(-g_j(\mathbf{x})\right)$$

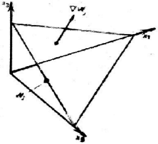

f
increasing c
f
g
feasible domain
x

Simplifications I

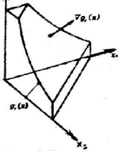
- They come from consideration of **practical problems** (contrary to the famous academic `three bar truss club` optimizations).

- **Experience from large-scale structural problems** proves that portion of compound failure surface **contributed by each failure function is small** and can be successfully linearized.

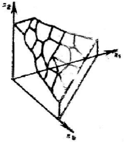
- Therefore, the **envelope of feasible designs** is transformed into a **piecewise linear failure surface**.



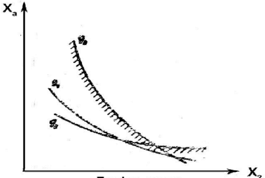
Weight surface



Min - Max Constraints
and Response constriction



Composite boundary
surface



Design space

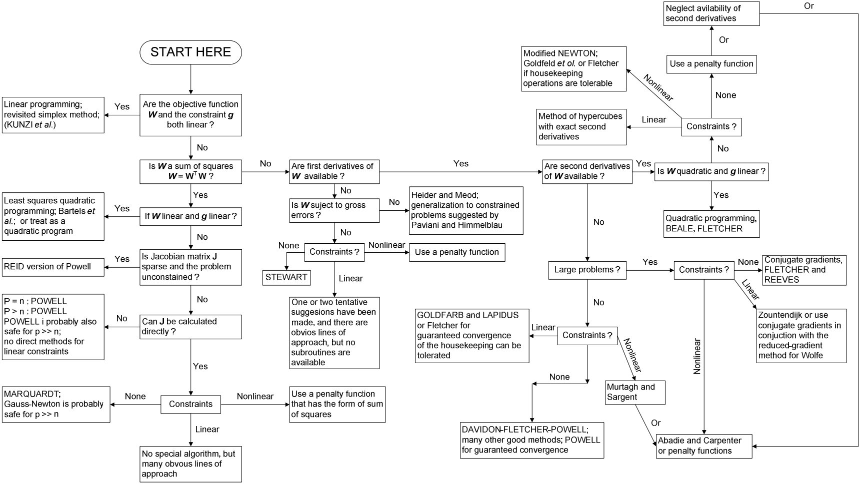
Simplifications II

- If the designer's **objective functions are monotonous** (only increasing or decreasing) with increase in structural scantlings the optimal designs would lie on that surface.

- Functions for weight or cost of labor and material usually have this feature.

- That practical (non academic) reasoning opens possibility to use simple and efficient methods of operations research, Hughes, Mistree, Zanic (1980).

OR menu of problem tailored methods (see Fletcher)



Design of experiments (FFE-see Montgomery)

	Intop					1	1	1	2	2	2	3	3	3			
	Bperc					1	2	3	1	2	3	1	2	3			
	Nloan					1	2	3	2	3	1	3	1	2			
Trial	L_{PP}	B_x	T_x	C_P	C_x	Membership grade for a combination of variables and parameters levels									σ	μ_e	S/N
1	1	1	1	1	1	0.868	0.907	0.903	0.941	0.892	0.745	0.945	0.839	0.823	0.064	0.874	22.76
2	1	1	2	2	2	0.620	0.645	0.660	0.710	0.648	0.513	0.715	0.594	0.578	0.064	0.631	19.86
3	1	1	3	3	3	0.406	0.422	0.433	0.468	0.424	0.337	0.472	0.388	0.378	0.043	0.414	19.72
4	1	2	1	2	2	0.783	0.807	0.822	0.867	0.810	0.665	0.872	0.755	0.739	0.065	0.791	21.70
5	1	2	2	3	3	0.525	0.546	0.559	0.601	0.548	0.438	0.606	0.503	0.491	0.053	0.535	20.04
6	1	2	3	1	1	0.795	0.819	0.834	0.878	0.822	0.677	0.883	0.767	0.751	0.065	0.803	21.87
7	1	3	1	3	3	0.715	0.738	0.753	0.797	0.741	0.608	0.802	0.689	0.674	0.061	0.724	21.44
8	1	3	2	1	1	0.944	0.959	0.967	0.987	0.961	0.854	0.989	0.926	0.914	0.042	0.944	27.01
9	1	3	3	2	2	0.743	0.766	0.781	0.825	0.769	0.634	0.830	0.717	0.702	0.062	0.752	21.72
10	2	1	1	2	3	0.612	0.636	0.652	0.700	0.640	0.508	0.705	0.587	0.572	0.062	0.623	19.99
11	2	1	2	3	1	0.623	0.647	0.662	0.711	0.650	0.517	0.716	0.597	0.582	0.063	0.634	20.04
12	2	1	3	1	2	0.643	0.668	0.683	0.732	0.671	0.535	0.738	0.616	0.601	0.064	0.654	20.14
13	2	2	1	3	1	0.782	0.806	0.820	0.865	0.809	0.666	0.870	0.754	0.738	0.064	0.790	21.83
14	2	2	2	1	2	0.789	0.813	0.828	0.871	0.816	0.673	0.876	0.762	0.746	0.064	0.797	21.93
15	2	2	3	2	3	0.540	0.560	0.574	0.616	0.563	0.451	0.621	0.518	0.505	0.054	0.549	20.20
16	2	3	1	1	2	0.954	0.967	0.974	0.991	0.968	0.869	0.992	0.937	0.926	0.039	0.953	27.86
17	2	3	2	2	3	0.727	0.750	0.764	0.808	0.753	0.621	0.812	0.701	0.687	0.060	0.736	21.73
18	2	3	3	3	1	0.750	0.773	0.787	0.830	0.776	0.643	0.835	0.725	0.710	0.060	0.759	21.98
19	3	1	1	3	2	0.623	0.647	0.663	0.711	0.651	0.519	0.716	0.597	0.583	0.063	0.634	20.12
20	3	1	2	1	3	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.00
21	3	1	3	2	1	0.633	0.657	0.673	0.721	0.661	0.527	0.726	0.607	0.592	0.063	0.644	20.17
22	3	2	1	1	3	0.769	0.793	0.807	0.852	0.796	0.655	0.857	0.741	0.726	0.064	0.777	21.73
23	3	2	2	2	1	0.777	0.801	0.815	0.860	0.804	0.662	0.864	0.750	0.734	0.064	0.785	21.83
24	3	2	3	3	2	0.547	0.568	0.581	0.624	0.571	0.458	0.628	0.525	0.512	0.054	0.557	20.29
25	3	3	1	2	1	0.945	0.959	0.967	0.987	0.961	0.858	0.988	0.927	0.916	0.041	0.945	27.31
26	3	3	2	3	2	0.733	0.756	0.770	0.813	0.758	0.628	0.818	0.708	0.693	0.060	0.741	21.85
27	3	3	3	1	3	0.734	0.757	0.771	0.814	0.760	0.629	0.819	0.709	0.695	0.060	0.743	21.87

Fractional factorial design around design with minimum L_∞