Linearna algebra

u MATLAB-u i R-u

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Računalna matematika
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NAPOMENA: Ova prezentacija dodatak je sadržaju pripadajućih ".m" i ".r" datoteka.

Pregled

1. Uvod
2. Matrice i vektori
    - Kreiranje matrica i vektora
    - Indeksiranje i odabiranje elemenata matrice
3. Osnovne operacije matrične algebre
4. Rang, slika i jezgra
5. Sustavi linearnih jednadžbi
6. Skalarni produkt
7. Vektorski produkt
8. Vektorske i matrične norme
9. Svojstvene vrijednosti
10. Singularnge vrijednosti

Linearna algebra

UVOD

Linearna algebra je dio matematike koji se bavi vektorskim prostorima i linarnim mapiranjima (funkcijama).

Zašto linearna algebra (kao tema u ovom kolegiju)?

- Zauzima centralno mjesto među matemačkim disciplinama.

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- Ključna uloga u gotovo svim znanostima koje koriste matematiku.

Linearna algebra
UVOD

Primjeri primjena

- inženjerstvo, tehničke i prirodne znanosti
  - teorija grafova, automatska regulacija, mehanika kontinuuma (...konačni elementi...), elektrotehnika, kemija, obada slika/signala, kodiranje...
- ekonomija, biologija,... općenito obrada podataka...
- optimizacija (inženjerstvo, financije, ekonomija,...)
- ... i još puno toga... gotovo svugdje gdje ima i matematike

Linearna algebra
UVOD

Linearnost

Neka su U, V linearni prostori (nad istim poljem F). Preslikavanje $f : U \to V$ je *linearno* ako ima ova svojstva

- Aditivnost $f(x+y) = f(x) + f(y), \quad \forall x, y \in U$
- Homogenost $f(\alpha x) = \alpha f(x), \quad \forall \alpha \in F$ i $\forall x \in U$,

$$Ax = y$$

$A \in \mathbb{R}^{n \times m}$ Linearno preslikavanje

$x \in \mathbb{R}^m$ Element (vektor, točka) linearnog prostora

$y \in \mathbb{R}^n$ Element (vektor, točka) linearnog prostora

Linearna algebra
UVOD

Mnogi nelinearni problemi (modeli) se aproksimiaju linearnim (modelima).

$$f(x_0 + \Delta x) = f(x_0) + D(x_0)\Delta x + \ldots \text{(Taylorov red)}$$

Neke "dobre strane" linearnosti

- superpozicija
- "lokalno je globalno" (npr. kod stabilnosti sustava)
- skalabilnost, numerika

Koliko je linearnost važna vidi se i iz toga koliko se često koristi pojam *nelinearno* (npr. nelinearni dinamički sustavi).

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Matrice i vektori

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & \dots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{pmatrix} \in \mathbb{R}^{n \times m}$$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n \quad \text{vektor stupac}$$

$$y = \begin{pmatrix} y_1 & y_2 & \dots & y_m \end{pmatrix} \in \mathbb{R}^{1 \times m} \quad \text{vektor redak}$$

Matrice i vektori

Kreiranje matrica i vektora u Matlabu i R-u

Description	MATLAB	R
Enter a row vector $\vec{v} = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}$	`v=[1 2 3 4]`	`v=c(1,2,3,4)` or alternatively `v=scan()` then enter "1 2 3 4" and press Enter twice (the blank line terminates input)
Enter a column vector $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$	`[1; 2; 3; 4]`	`c(1,2,3,4)` (R does not distinguish between row and column vectors.)
Enter a matrix $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$	`[1 2 3 ; 4 5 6]`	To enter values by row: `matrix(c(1,2,3,4,5,6), nrow=2, byrow=TRUE)` To enter values by column: `matrix(c(1,4,2,5,3,6), nrow=2)`

Napomena: za više detalja i primjere vidjeti pripadajuće ".m" i ".r" datoteke i podloge za vježbe.

Matrice i vektori

Kreiranje matrica i vektora u Matlabu i R-u

Description	MATLAB	R
Build the vector [2 3 4 5 6 7]	`2:7`	`2:7`
Build the vector [7 6 5 4 3 2]	`7:-1:2`	`7:2`
Build the vector [2 5 8 11 14]	`2:3:14`	`seq(2,14,3)`
Build a vector of length k containing all zeros	`zeros(k,1)` (for a column vector) or `zeros(1,k)` (for a row vector)	`rep(0,k)`
Build a vector of length k containing the value j in all positions	`j*ones(k,1)` (for a column vector) or `j*ones(1,k)` (for a row vector)	`rep(j,k)`
Build an $m \times n$ matrix of zeros	`zeros(m,n)`	`matrix(0,nrow=m,ncol=n)` or just `matrix(0,m,n)`
Build an $m \times n$ matrix containing j in all positions	`j*ones(m,n)`	`matrix(j,nrow=m,ncol=n)` or just `matrix(j,m,n)`
$n \times n$ identity matrix I_n	`eye(n)`	`diag(n)`
Build diagonal matrix A using elements of vector **v** as diagonal entries	`diag(v)`	`diag(v,nrow=length(v))` (Note: if you are sure the length of vector **v** is 2 or more, you can simply say `diag(v)`.)
Extract diagonal elements of matrix A	`v=diag(A)`	`v=diag(A)`
"Glue" two matrices `a1` and `a2` (with the same number of rows) side-by-side	`[a1 a2]`	`cbind(a1,a2)`
"Stack" two matrices `a1` and `a2` (with the same number of columns) on top of each other	`[a1; a2]`	`rbind(a1,a2)`

Gdje smo:

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3 Osnovne operacije matrične algebre

4 Rang, slika i jezgra

5 Sustavi linearnih jednadžbi

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Matrice i vektori

Indeksiranje i odabiranje elemenata matrice

Description	MATLAB	R
Access an element of vector **v**	`v(3)`	`v[3]`
Access an element of matrix **A**	`A(2,3)`	`A[2,3]`
Access an element of matrix **A** using a single index: indices count down the first column, then down the second column, etc.	`A(5)`	`A[5]`
Reverse the order of elements in vector **v**	`v(end:-1:1)`	`rev(v)`
Column 2 of matrix **A**	`A(:,2)`	`A[,2]` Note: that gives the result as a vector. To make the result a $m \times 1$ matrix instead, do `A[,2,drop=FALSE]`
Row 7 of matrix **A**	`A(7,:)`	`A[7,]` Note: that gives the result as a vector. To make the result a $1 \times n$ matrix instead, do `A[7,,drop=FALSE]`
All elements of **A** as a vector, column-by-column	`A(:)` (gives a column vector)	`c(A)`
Rows 2–4, columns 6–10 of **A** (this is a 3×5 matrix)	`A(2:4,6:10)`	`A[2:4,6:10]`
A 3×2 matrix consisting of rows 7, 7, and 6 and columns 2 and 1 of A (in that order)	`A([7 7 6], [2 1])`	`A[c(7,7,6),c(2,1)]`
Circularly shift the rows of matrix A down by s_1 elements, and right by s_2 elements	`circshift(A, [s1 s2])`	No simple way, but modulo arithmetic on indices will work: `m=dim(A)[1]; n=dim(A)[2]; A[(1:m-s1-1)%%m+1, (1:n-s2-1)%%n+1]`

Osnovne operacije matrične algebre

Description	MATLAB	R
Vector dot product $\vec{x} \cdot \vec{y} = \vec{x}^T \vec{y}$	`dot(x,y)`	`sum(x*y)`
Vector cross product $\vec{x} \times \vec{y}$	`cross(x,y)`	Not in base R, but e.g. the **xprod** function from the RSEIS package will do it (see item 331 for how to install/load packages)
Matrix multiplication AB	`A * B`	`A %*% B`
Element-by-element multiplication of A and B	`A .* B`	`A * B`
Transpose of a matrix, A^T	`A'` (This is actually the complex conjugate (i.e. Hermitian) transpose; use `A.'` for the non-conjugate transpose if you like; they are equivalent for real matrices.)	`t(A)` for transpose, or `Conj(t(A))` for conjugate (Hermitian) transpose
Solve $A\vec{x} = \vec{b}$	`A\b` Warning: if there is no solution, MATLAB gives you a least-squares "best fit." If there are many solutions, MATLAB just gives you one of them.	`solve(A,b)` Warning: this only works with square invertible matrices.
Reduced echelon form of A	`rref(A)`	R does not have a function to do this
Determinant of $\mathbf{A}$	`det(A)`	`det(A)`
Inverse of $\mathbf{A}$	`inv(A)`	`solve(A)`
Trace of $\mathbf{A}$	`trace(A)`	`sum(diag(A))`
Compute AB^{-1}	`A/B`	`A %*% solve(B)`
Element-by-element division of A and B	`A ./ B`	`A / B`
Compute $A^{-1}B$	`A\B`	`solve(A,B)`

Sustavi linearnih jednadžbi

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1m}x_m &= b_1, \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2m}x_m &= b_2, \\
\vdots \qquad\qquad\qquad\quad \vdots \qquad & \quad \vdots \\
a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nm}x_m &= b_n
\end{aligned}
$$

$$
\begin{pmatrix}
a_{11} & a_{12} & \ldots & a_{1m} \\
a_{21} & a_{22} & \ldots & a_{2m} \\
\vdots & \vdots & \ldots & \vdots \\
a_{n1} & a_{n2} & \ldots & a_{nm}
\end{pmatrix}
\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}
=
\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}
$$

$$
Ax = b
$$

Slika, jezgra i rang

$$A \in \mathbb{R}^{m \times n}$$

Slika (*eng. image (space), range*)

$$\mathrm{Im}(A) = \mathrm{Range}(A) = \mathcal{R}(A) = \{\, Ax \mid x \in \mathbb{R}^n \} \subseteq \mathbb{R}^m$$

Jezgra (*eng. kernel (space), null space*)

$$\mathrm{Ker}(A) = \mathrm{Null}(A) = \mathcal{N}(A) = \{\, x \in \mathbb{R}^n \mid Ax = 0 \,\}$$

Rang matrice

$$\mathrm{Rang}(A) = \dim(A)$$

Slika

$$A \in \mathbb{R}^{m \times n}$$

Slika (*eng. image (space), range*)

$$\mathrm{Im}(A) = \mathrm{Range}(A) = \mathcal{R}(A) = \{ Ax \mid x \in \mathbb{R}^n \} \subseteq \mathbb{R}^m$$

$\mathcal{R}(A)$ se može interpretirati kao:

- prostor koji rasprostiru stupci matrice A
- skup vektora y za koje jednadžba $Ax = y$ ima rješenje

MATLAB: Traženje ortonormalne baze prostora slike matrice A:
`orth(A)`

Jezgra

$$A \in \mathbb{R}^{m \times n}$$

Jezgra (*eng. kernel (space), null space*)

$$\mathrm{Ker}(A) = \mathrm{Null}(A) = \mathcal{N}(A) = \{\, x \in \mathbb{R}^n \mid Ax = 0 \,\}$$

- $\mathcal{N}(A)$ je skup vektora u $\mathbb{R}^n$ okomitih na redke matrice A
- Ako je $y = Ax$ i $z \in \mathcal{N}(A)$, tada je $y = A(x + z)$
- Ako je $y = Ax$ i $y = A\tilde{x}$, tada je $\tilde{x} = x + z$ za neki $z \in \mathcal{N}(A)$

MATLAB: Traženje ortonormalne baze prostora jezgre matrice A:
`null(A)`

Rang matrice

$$A \in \mathbb{R}^{m \times n}$$

Rang matrice

$$\mathrm{Rang}(A) = \dim(A)$$

- $\mathrm{Rang}(A) + \dim(\mathcal{N}(A)) = n$
- $\mathrm{Rang}(A)$ je maksimalan broj linearno nezavisnih stupaca (= broj nezavisnih redaka) u A
- $\mathrm{Rang}(A) = \mathrm{Rang}(A^\top)$

MATLAB: `rank(A)`, `rank(A,tol)` (Oprez: numerika)
R: `qr(A)$rank`

Sustavi linearnih jednadžbi

$$
\begin{aligned}
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\vdots \qquad\qquad\qquad\qquad \vdots \quad &\quad\ \vdots \\
a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nm}x_m &= b_n
\end{aligned}
$$

$$
\begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1m} \\ a_{21} & a_{22} & \ldots & a_{2m} \\ \vdots & \vdots & \ldots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nm} \end{pmatrix}
\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}
=
\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}
$$

$$Ax = b$$

Sustavi linearnih jednadžbi

Description	MATLAB	R
Solve $A\vec{x} = \vec{b}$	`A\b` Warning: if there is no solution, MATLAB gives you a least-squares "best fit." If there are many solutions, MATLAB just gives you one of them.	`solve(A,b)` Warning: this only works with square invertible matrices.

MATLAB: $A \setminus B$ is the matrix division of A into B, which is roughly the same as INV(A)*B , except it is computed in a different way. If A is an N-by-N matrix and B is a column vector with N components, or a matrix with several such columns, then $X = A \setminus B$ is the solution to the equation A*X = B. A warning message is printed if A is badly scaled or nearly singular. $A \setminus EYE(SIZE(A))$ produces the inverse of A.

Sustavi linearnih jednadžbi

MATLAB: If A is an M-by-N matrix with M < or > N and B is a column vector with M components, or a matrix with several such columns, then $X = A \setminus B$ is the solution in the least squares sense to the under- or overdetermined system of equations $A * X = B$. The effective rank, K, of A is determined from the QR decomposition with pivoting. A solution X is computed which has at most K nonzero components per column. If K < N this will usually not be the same solution as PINV(A)*B. $A \setminus EYE(SIZE(A))$ produces a generalized inverse of A.

Skalarni produkt vektora

$$x, y \in \mathbb{R}^n$$

Skalarni produkt (*Inner product*)

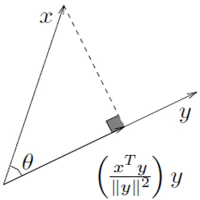
$$\langle x, y \rangle = x_1 y_1 + x_2 y_2 + \ldots + x_n y_n = x^\top y$$

Description	MATLAB	R
Vector dot product $\vec{x} \cdot \vec{y} = \vec{x}^T \vec{y}$	`dot(x,y)`	`sum(x*y)`

Cauchy-Schwartz nejednakost i kut između vektora

Za svaki $x, y \in \mathbb{R}^n$ vrijedi

$$|x^\top y| \leq \|x\| \, \|y\| \qquad \text{(Cauchy-Schwartz)}$$



Kut imeđu vektora u $\mathbb{R}^n$:

$$\theta = \angle(x, y) = \cos^{-1} \frac{x^\top y}{\|x\| \, \|y\|}$$

$$x^\top y = \|x\| \, \|y\| \cos\theta$$

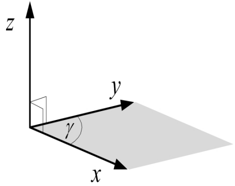
Vektorski produkt

$$x = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^\top \in \mathbb{R}^3, \quad y = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix}^\top \in \mathbb{R}^3$$

Vektorski produkt

$$z = x \times y = \begin{bmatrix} x_2 y_3 - x_3 y_2 \\ x_3 y_1 - x_1 y_3 \\ x_1 y_2 - x_2 y_1 \end{bmatrix}$$

Description	MATLAB	R
Vector cross product $\vec{x} \times \vec{y}$	`cross(x,y)`	Not in base R, but e.g. the **xprod** function from the RSEIS package will do it (see item 331 for how to install/load packages)



$$\|z\| = \|x\| \, \|y\| \sin\gamma$$

Vektorske norme

Norma je sklarna vrijednost koja služi kao mjera veličine vektora ili matrice.
Tzv. p-norma vektora $x \in \mathbb{R}^n(\mathbb{C}^n)$ definarana je

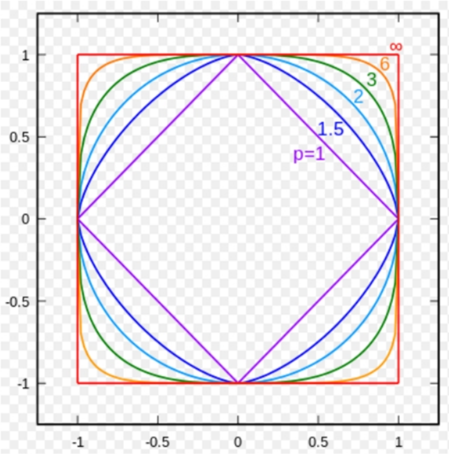
$$\|x\|_p := \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}}, \qquad 1 \leq p < \infty,$$

$$\|x\|_\infty := \max_{1 \leq i \leq n} (|x_i|), \qquad p = \infty,$$

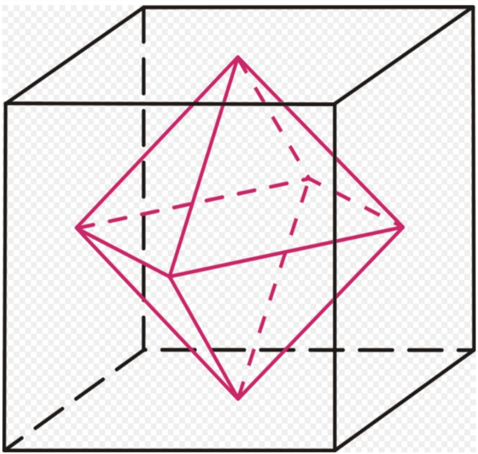
Euklidska norma (2-norma):

$$\|x\|_2 = \sqrt{x^* x} = \sqrt{x_1^2 + x_2^2 + \ldots x_n^2}$$

Vektorske norme



Vektorske norme



Vektorske i matrične norme

Inducirane matrične norme

$$\|A\|_p := \max_{x \neq 0} \frac{\|Ax\|_p}{\|x\|_p}$$

Za $A \in \mathbb{R}^{m \times n}$

$$\|A\|_1 = \max_{1 \leq j \leq n} \sum_{i=1}^{m} |a_{ij}|$$

$$\|A\|_2 = \sqrt{\lambda_{max}(A^* A)} = \sigma_{max}(A)$$

$$\|A\|_\infty = \max_{1 \leq j \leq m} \sum_{i=1}^{n} |a_{ij}|$$

Vektorske i matrične norme

Description	MATLAB	R
Vector norms	`norm(v,1)` for 1-norm $\|\vec{v}\|_1$, `norm(v,2)` for Euclidean norm $\|\vec{v}\|_2$, `norm(v,inf)` for infinity-norm $\|\vec{v}\|_\infty$, and `norm(v,p)` for p-norm $\|\vec{v}\|_p = (\sum \|v_i\|^p)^{1/p}$	R does not have a **norm** function for vectors; only one for matrices. But the following will work: `norm(matrix(v),'1')` for 1-norm $\|\vec{v}\|_1$, `norm(matrix(v),'i')` for infinity-norm $\|\vec{v}\|_\infty$, and `sum(abs(v)^p)^(1/p)` for p-norm $\|\vec{v}\|_p = (\sum \|v_i\|^p)^{1/p}$
Matrix norms	`norm(A,1)` for 1-norm $\|A\|_1$, `norm(A)` for 2-norm $\|A\|_2$, `norm(A,inf)` for infinity-norm $\|A\|_\infty$, and `norm(A,'fro')` for Frobenius norm $\left(\sum_i (A^T A)_{ii}\right)^{1/2}$	`norm(A,'1')` for 1-norm $\|A\|_1$, `max(svd(A)$d)` for 2-norm $\|A\|_2$, `norm(A,'i')` for infinity-norm $\|A\|_\infty$, and `norm(A,'f')` for Frobenius norm $\left(\sum_i (A^T A)_{ii}\right)^{1/2}$

Svojstvene vrijednosti

Neka je $A \in \mathbb{C}^{n \times n}$. Skalar $\lambda \in \mathbb{C}$ zove se *svojstvena vrijednost* matrice A, ako postoji vektor $x \in \mathbb{C}^n$, $x \neq 0$, takav da je

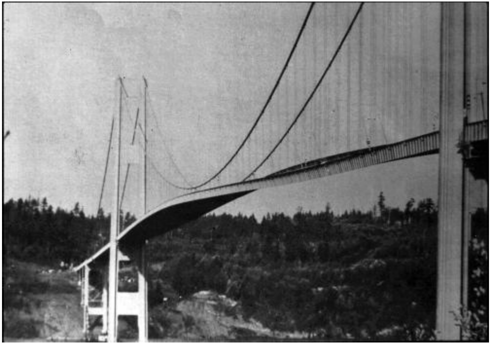
$$Ax = \lambda x.$$

Takav vektor x zove se *svojstveni vektor* od A, koji pripada svojstvenoj vrijednosti λ.

Description	MATLAB	R
Set **w** to be a vector of eigenvalues of **A**, and **V** a matrix containing the corresponding eigenvectors	`[V,D]=eig(A)` and then `w=diag(D)` since MATLAB returns the eigenvalues on the diagonal of **D**	`tmp=eigen(A); w=tmp$values; V=tmp$vectors`

Svojstvene vrijednosti

Primjer: ponašanje dinamičkih sustava
Vibracije Tacoma Narrows mosta



Frekvencija uzbudnih sila vjetra bila je blizu rezonantnoj frekvanciji mosta (svojstvena vrijednost!)
Vidjeti impresivan video na Youtubeu (Tacoma Narrows Bridge)

Svojstvene vrijednosti

Primjer: ponašanje dinamičkih sustava
Vibracije Tacoma Narrows mosta

Svojstvene vrijednosti

Primjer: ponašanje dinamičkih sustava

Skalarna linearna diferecijalna jednadžba

$$\frac{dx}{dt} = \lambda x, \quad x(0) = x_0$$

Rješenje?

Svojstvene vrijednosti

Primjer: ponašanje dinamičkih sustava

Skalarna linearna diferecijalna jednadžba

$$\frac{dx}{dt} = \lambda x, \quad x(0) = x_0$$

Rješenje?

$$x(t) = e^{\lambda t} x_0$$

Ravnotežna točka $x_e = 0$ je

- stabilna ako je $\lambda \leq 0$
- asimptotski stabilna ako je $\lambda < 0$

Primjer: ponašanje dinamičkih sustava

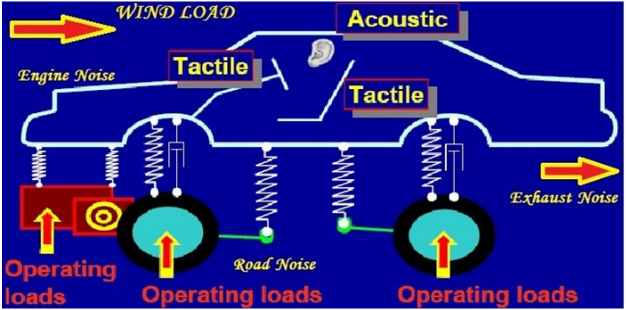
Sustav linearnih jednadžbi, za $x(t) \in \mathbb{R}^n$, $A \in \mathbb{R}^{n \times n}$:

$$\frac{dx}{dt} = Ax, \quad x(0) = x_0$$

Ravnotežna točka $x_e = 0$ je asimptotski stabilna ako je $Re(\lambda_i) < 0$ za svaki $i = 1, \ldots, n$; gdje λ_i označava i-tu svojstvenu vrijednost matrice A, a Re označava realni dio kompleksnog broja.

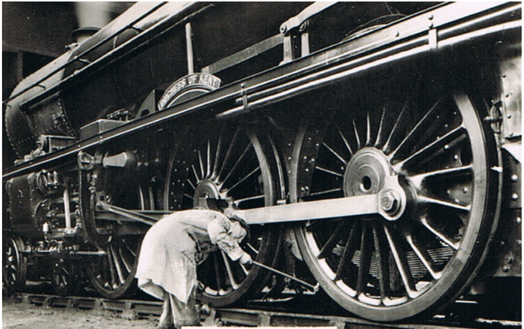
Svojstvene vrijednosti

Primjer: konstrukcija automobila. Svojstvene vrijednosti = vibracije, udobnost vožnje



Svojstvene vrijednosti

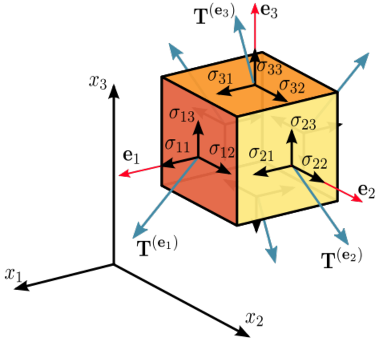
Primjer: provjera konstrukcije na pukotine i deformacije.



THE WHEEL TAPPER

Ne treba provjeravati svaki centimetar konstrukcije (neučinkovito); već konstrukciju u cjelini kroz svojstvene vrijednosti (učinkovito).
Svojstvene vrijednosti = vibracije, zvuk. Pukotine $\Longrightarrow$ promjena svj. vrijednosti

Svojstvene vrijednosti



$$\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$

Svojstvene vrijednosti tenzora naprezanja σ su glavna naprezanja.
Pripadajući svojstveni vektori definiraju smjerove glavnih naprezanja.

Singularne vrijednosti

Za svaku matricu $A \in \mathbb{C}^{m \times n}$ postoji *dekompozicija singularnih vrijednosti*

$$A = UDV^*$$

gdje je

$$\begin{aligned} U &\in \mathbb{C}^{m \times m} \quad \text{unitarna matrica,} \quad U^*U = UU^* = I \\ V &\in \mathbb{C}^{n \times n} \quad \text{unitarna matrica,} \quad V^*V = VV^* = I \\ D &\in \mathbb{C}^{m \times n} \quad \text{sa dijagonalnim elemetima } \sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_p \geq 0, \\ & \quad p = \min(m, n) \end{aligned}$$

Singularne vrijednosti

Za svaku matricu $A \in \mathbb{C}^{m\times n}$ postoji *dekompozicija singularnih vrijednosti*

$$A = UDV^*$$

gdje je

$$D \in \mathbb{C}^{m\times n} \quad \text{sa dijagonalnim elemetima } \sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_p \geq 0$$

$$\text{Za } m \neq n: \qquad \Sigma = \begin{bmatrix} \sigma_1 & 0 & \ldots & 0 \\ 0 & \sigma_2 & & 0 \\ & & \ddots & \\ 0 & & & \sigma_n \\ 0 & \ldots & & 0 \\ \vdots & & & \vdots \\ 0 & \ldots & & 0 \end{bmatrix} \quad \text{ili} \quad \Sigma = \begin{bmatrix} \sigma_1 & 0 & \ldots & 0 & 0 & \ldots & 0 \\ 0 & \sigma_2 & & & \vdots & & \vdots \\ & & \ddots & & & & \\ 0 & \ldots & & \sigma_m & 0 & \ldots & 0 \end{bmatrix}$$

Singularne vrijednosti

Za svaku matricu $A \in \mathbb{C}^{m \times n}$ postoji *dekompozicija singularnih vrijednosti*

$$A = UDV^*$$

gdje su $U \in \mathbb{C}^{m \times m}$, $V \in \mathbb{C}^{n \times n}$ unitarne matrice i $\Sigma \in \mathbb{C}^{m \times n}$ je dijagonalna matrica.

- brojevi $\sigma_1, \sigma_2, \ldots, \sigma_p$ su singularne vrijednosti od A
- $U = (u_1, \ldots, u_m)$; vektori $u_i \in \mathbb{C}^m$ su lijevi singularni vektori od A
- $V = (v_1, \ldots, v_n)$; vektori $v_i \in \mathbb{C}^n$ su desni singularni vektori od A
- broj singularnih vrijednosti p različitih od nule jednak je ragu matrice A

Singularne vrijednosti

$$A = \underbrace{\left[u_1 \mid u_2 \mid \ldots \mid u_m \right]}_{U \in \mathbb{C}^{m \times m}} \underbrace{\begin{bmatrix} \sigma_1 & 0 & \ldots & 0 \\ 0 & \sigma_2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & 0 & \sigma_n \\ 0 & & 0 & 0 \\ \vdots & & & \vdots \\ 0 & \ldots & \ldots & 0 \end{bmatrix}}_{\Sigma \in \mathbb{R}^{m \times n}} \underbrace{\begin{bmatrix} v_1^* \\ \hline v_2^* \\ \hline \vdots \\ \hline v_n^* \end{bmatrix}}_{V^* \in \mathbb{C}^{n \times n}}$$

$$Av_1 = U \Sigma V^* v_1 = U \Sigma \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = U \begin{bmatrix} \sigma_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \sigma_1 u_1$$

$$Av_2 = U \Sigma V^* v_2 = U \Sigma \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} = U \begin{bmatrix} 0 \\ \sigma_2 \\ \vdots \\ 0 \end{bmatrix} = \sigma_2 u_2$$

Singularne vrijednosti

$$A = \underbrace{\left[u_1 \mid u_2 \mid \dots \mid u_m \right]}_{U \in \mathbb{C}^{m\times m}} \underbrace{\begin{bmatrix} \sigma_1 & 0 & \dots & 0 \\ 0 & \sigma_2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & 0 & \sigma_n \\ 0 & & 0 & 0 \\ \vdots & & & \vdots \\ 0 & \dots & \dots & 0 \end{bmatrix}}_{\Sigma \in \mathbb{R}^{m\times n}} \underbrace{\begin{bmatrix} v_1^* \\ \hline v_2^* \\ \hline \vdots \\ \hline v_n^* \end{bmatrix}}_{V^* \in \mathbb{C}^{n\times n}}$$

$$Av_1 = U\Sigma V^* v_1 = U\Sigma \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = U \begin{bmatrix} \sigma_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \sigma_1 u_1$$

$$\|A\| = \max_{\|x\| \neq 0} \frac{\|Ax\|}{\|x\|} = \sigma_{\max}(A) = \sigma_1(A)$$

Najveća singularna vrijdnost matrice je ujedno i norma te matrice

Singularne vrijednosti

Description	MATLAB	R
Singular-value decomposition: given $m \times n$ matrix A with rank r, find $m \times r$ matrix P with orthonormal columns, diagonal $r \times r$ matrix S, and $r \times n$ matrix Q^T with orthonormal rows so that $PSQ^T = A$	`[P,S,Q]=svd(A,'econ')`	`tmp=svd(A); U=tmp$u; V=tmp$v; S=diag(tmp$d)`